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The Mellin Transform in Signal Analysis
Technical Report under Grant N00014-91-J-4138

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A review of the Fourier, Mellin, and Hilbert transforms is provided to lay the foundation for developing a new transform denoted the K-transform: a composition of the Fourier and Mellin transforms. In this report, the transform is defined, an explicit expression as an integral operator is derived, and an asymptotic estimate for the transform kernel is obtained. Some properties of the K-transform are explored, and the application of this transform to the work of Altes on mammalian hearing is noted. Lastly, a unified setting for Fourier and wavelet analysis is explored. The connection of the Heisenberg group with the Gabor transform and Wigner-Ville distribution is shown, as is the connection of the affine group with the wavelet transform. These notions are then combined in a single setting, the affine-Heisenberg group. In this setting, the report is closed by introducing an affine version of the Wigner-Ville distribution in terms of the K-transform.

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EXECUTIVE SUMMARY

The Fourier transform has long been a standard tool in signal analysis. It performs the fundamental task of analyzing the frequencies which exist in a given function of time. The Fourier transform does not, however, allow one to discern at what time particular frequencies exist in a signal. To overcome this predicament, Gabor introduced the short-time Fourier transform which permits a "time-frequency analysis" of a given signal; that is, it provides a means of telling both *what* frequencies exist in a signal as well as *when* they exist. This time-frequency analysis led to the development of the Wigner-Ville distribution and its connection with the Heisenberg group.

More recently, the desire to analyze signals at different levels of resolution has led to the notion of *scale* in signal analysis. While the analysis performed by the Gabor transform is "fixed scale," scaling plays a fundamental role in the emerging field of wavelet theory. This emphasis on dilation naturally leads to the consideration of the affine group and the Mellin transform.

In this report, we begin with a review of the Fourier transform and some of its characteristics. We then introduce the Mellin transform, showing its relationship to the Fourier transform and developing its analogous properties. In the following section, the Paley-Wiener theorem is stated and proved, and the Hilbert transform is introduced. In particular, conditions which restrict the support of a function's Fourier transform are explored.

With this background, the stage is set for the introduction of the K-transform as the composition of the Fourier and Mellin transforms. We define this transformation, derive its explicit expression as an integral operator, and obtain an asymptotic estimate for the transform kernel. Some properties of the K-transform are explored, and the application of this transform to the work of Altes on mammalian hearing is noted.

In the final section, we develop a unified setting for Fourier and wavelet analysis. The connection of the Heisenberg group with the Gabor transform and Wigner-Ville distribution is shown, as is the connection of the affine group with the wavelet transform. We then combine these notions in a single setting, the affine-Heisenberg group. In this setting, we close by introducing an affine version of the Wigner-Ville distribution in terms of the K-transform.

1. FOURIER AND MELLIN TRANSFORMS

For a finite energy signal, a function $f = f(x)$ in the space $L^2(\mathbb{R})$ of all square-integrable functions on $\mathbb{R}$, its Fourier transform $\mathcal{F}(f) = \hat{f}$ is defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx \tag{1.1}$$

and its Fourier integral representation by

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}\,d\xi. \tag{1.2}$$

Strictly speaking, for L^2-functions these integrals must be interpreted as the limit

$$\hat{f}(\xi) = \lim_{n\to\infty} \int_{-n}^{n} f(x)e^{-2\pi i x\xi}\,dx$$

taken in norm (analogously for the inverse Fourier transform). Plancherel's theorem

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\,d\xi \quad = \quad \int_{-\infty}^{\infty} |f(x)|^2\,dx \tag{1.3}$$

for Fourier transforms ensures that $\mathcal{F}$ is an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$; on the other hand, Parseval's theorem

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \quad = \quad \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi \tag{1.4}$$

for Fourier transforms ensures that $\mathcal{F}$ is a unitary mapping from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

The Fourier transform is a fundamental tool in signal analysis. By using a complex exponential as the transform kernel, this analysis allows the decomposition of a signal into its individual frequency components and establishes the relative intensity of each component.[1] In mathematical discussions of the Fourier transform, time and frequency are often distinguished by writing $\mathbb{R}$ for the time domain and $\hat{\mathbb{R}}$ for the frequency domain; also we shall usually use Roman letters $x, t, \ldots$ for the time variable and Greek letters $\xi, \eta, \ldots$ for the frequency variable. Whether on the time or frequency side, however, the Fourier transform exploits the additive property of real numbers through use of the fundamental property $e^{2\pi i(x+y)} = e^{2\pi i x}e^{2\pi i y}$ in establishing the convolution property

$$\mathcal{F} : f * g \longrightarrow \mathcal{F}\Big(\int_{-\infty}^{\infty} f(x-v)g(v)\,dv\Big) = (\mathcal{F}f)(\mathcal{F}g) \tag{1.5}$$

as well as the properties

$$\mathcal{F} : f(. + s) \longrightarrow e^{2\pi i s\xi}\mathcal{F}f \tag{1.6}$$

$$\mathcal{F} : e^{-2\pi i x\eta}f \longrightarrow (\mathcal{F}f)(\xi + \eta) \tag{1.7}$$

describing the interchange of translation and modulation. A simple change of variables yields the property

$$\mathcal{F} : f(r.) \longrightarrow \frac{1}{r}(\mathcal{F}f)(\frac{\xi}{r}) \tag{1.8}$$

describing the effect of dilation. Two other useful results,

$$\mathcal{F} : f' \longrightarrow 2\pi i\xi\mathcal{F}f \tag{1.9}$$

and

$$\mathcal{F} : xf \longrightarrow \frac{-1}{2\pi i}(\mathcal{F}f)', \tag{1.10}$$

follow immediately from integration by parts. Lastly,

$$\mathcal{F} : f^* \longrightarrow \overline{\mathcal{F}f}, \tag{1.11}$$

where $f^*(x) = \overline{f(-x)}$. In order to see the action of $\mathcal{F}$ on some specific examples, let us consider two very important functions, the Gaussian kernel and the Poisson kernel.

Gaussian kernel

$$\mathcal{F} : e^{-\pi x^2} \longrightarrow e^{-\pi\xi^2} \tag{1.12)(i}$$

We see this fact by completing the square in the exponent of the integrand:

$$\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi i x\xi}\,dx = \int_{-\infty}^{\infty} e^{-\pi(x^2+2ix\xi-\xi^2)}e^{-\pi\xi^2}\,dx = \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}e^{-\pi\xi^2}\,dx$$

$$= \int_{i\xi-\infty}^{i\xi+\infty} e^{-\pi y^2}e^{-\pi\xi^2}\,dy = e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi y^2}\,dy = e^{-\pi\xi^2}$$

since $\int_{-\infty}^{\infty} e^{-\pi y^2}\,dy = 1$. The change in the limits of integration $i\xi \pm \infty \longrightarrow \pm\infty$ is readily verified by applying Cauchy's Theorem to the contour integral around the box in $\mathbb{C}$ with vertices $-R, R, R + i\xi, -R + i\xi$, and then taking the limit as $R \longrightarrow \infty$.

Poisson kernel

$$\mathcal{F} : \frac{1}{\pi}\left(\frac{1}{1+x^2}\right) \longrightarrow e^{-2\pi|\xi|} \tag{1.12)(ii}$$

Here we observe that $f(x) = 1/\pi(1 + x^2)$ is a rational function whose denominator is two degrees greater than the numerator, so we may apply the Residue Theorem for rational functions (from complex analysis)[2] and exploit the fact that f is even:

$$\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx = \int_{-\infty}^{\infty} f(x)\cos 2\pi|\xi|x\,dx = \mathrm{Re}[2\pi i Res(i, f(z)e^{2\pi i|\xi|z})]$$

since i is the only pole of f in $\mathbb{C}_+$. Now calculate the residue

$$\operatorname{Res}(i, f(z)e^{2\pi i|\xi|z}) = \lim_{z\to i} \frac{(z-i)e^{2\pi i|\xi|z}}{\pi(1+z^2)} = \lim_{z\to i} \frac{e^{2\pi i|\xi|z}}{\pi(z+i)} = \frac{e^{-2\pi|\xi|}}{2\pi i}.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx = e^{-2\pi|\xi|}.$$

Let us now introduce the Mellin transform. Set

$$\xi_+ = \max(\xi, 0) \quad , \quad \xi_- = \max(-\xi, 0) \qquad (\xi \in \mathbb{R}) \tag{1.13}$$

and define Mellin transforms

$$\mathcal{M}_+\phi(t) = \int_{-\infty}^{\infty} \phi(\xi)\xi_+^{2\pi i t - 1/2}\,d\xi, \qquad \mathcal{M}_-\phi(t) = \int_{-\infty}^{\infty} \phi(\xi)\xi_-^{2\pi i t-1/2}\,d\xi. \tag{1.14}$$

Observe these are simply a slight modification of the "classical" definition [3]

$$\mathcal{M}\phi(z) = \int_0^{\infty} \phi(\xi)\xi^{z-1}\,d\xi, \tag{1.15}$$

where we have set $z = 2\pi i t + 1/2$ to make $\mathcal{M}_+\phi$ and $\mathcal{M}_-\phi$ functions of $t \in \mathbb{R}$. Also note the obvious relationship $\mathcal{M}_+[\phi(-\xi)] = \mathcal{M}_-\phi$. The corresponding inverse Mellin transforms are given by

$$\int_{-\infty}^{\infty} \mathcal{M}_+\phi(t)\overline{\xi_+^{2\pi i t-1/2}}\,dt = \begin{cases} \phi(\xi) & (\xi > 0), \\ 0 & (\xi \le 0); \end{cases} \tag{1.16}$$

and

$$\int_{-\infty}^{\infty} \mathcal{M}_-\phi(t)\overline{\xi_-^{2\pi i t-1/2}}\,dt = \begin{cases} \phi(\xi) & (\xi < 0), \\ 0 & (\xi \ge 0). \end{cases} \tag{1.17}$$

Now we point out the fundamental relationship between the Fourier and Mellin transforms:

$$\begin{aligned} \mathcal{F}^{-1} : e^{\xi/2}\phi(e^{\xi}) &\longrightarrow \int_{-\infty}^{\infty} \phi(e^{\xi})e^{\xi/2}e^{2\pi i\xi t}\,d\xi \\ &= \int_0^{\infty} \phi(\xi)\xi^{1/2}\xi^{2\pi i t}\,\frac{d\xi}{\xi} \end{aligned}$$

(by mapping $\xi \longrightarrow \log\xi$). Thus,

$$\mathcal{F}^{-1} : e^{\xi/2}\phi(e^{\xi}) \longrightarrow \int_{-\infty}^{\infty} \phi(\xi)\xi_+^{2\pi i t-1/2}\,d\xi = \mathcal{M}_+\phi(t); \tag{1.18}$$

similarly,

$$\mathcal{F}^{-1} : e^{\xi/2}\phi(-e^{\xi}) \longrightarrow \int_{-\infty}^{\infty} \phi(\xi)\xi_-^{2\pi i t-1/2}\,d\xi = \mathcal{M}_-\phi(t). \tag{1.19}$$

From this relationship we readily establish the analogue of Plancherel's theorem for the Mellin transforms:

$$\int_{-\infty}^{\infty} |\phi(\xi)|^2 \, d\xi = \int_{0}^{\infty} |\phi(\xi)|^2 \, d\xi + \int_{-\infty}^{0} |\phi(\xi)|^2 \, d\xi$$

$$= \int_{-\infty}^{\infty} |e^{\xi/2}\phi(e^{\xi})|^2 \, d\xi + \int_{-\infty}^{\infty} |e^{\xi/2}\phi(-e^{\xi})|^2 \, d\xi$$

$$= \int_{-\infty}^{\infty} |[e^{\xi/2}\phi(e^{\xi})]^{\vee}|^2 \, dt + \int_{-\infty}^{\infty} |[e^{\xi/2}\phi(-e^{\xi})]^{\vee}|^2 \, dt$$

by Plancherel's theorem for the Fourier transform. Thus, from Eqs. (1.18) and (1.19),

$$\int_{-\infty}^{\infty} |\phi(\xi)|^2 \, d\xi = \int_{-\infty}^{\infty} |\mathcal{M}_{+}\phi(t)|^2 \, dt + \int_{-\infty}^{\infty} |\mathcal{M}_{-}\phi(t)|^2 \, dt. \qquad (1.20)$$

In the same way we derive the analogue of Parseval's theorem

$$\int_{-\infty}^{\infty} \phi(\xi)\overline{\psi(\xi)} \, d\xi = \int_{-\infty}^{\infty} \mathcal{M}_{+}\phi(t)\overline{\mathcal{M}_{+}\psi(t)} \, dt + \int_{-\infty}^{\infty} \mathcal{M}_{-}\phi(t)\overline{\mathcal{M}_{-}\psi(t)} \, dt. \qquad (1.21)$$

In relating the Fourier transform to the Mellin transform, it is the mapping $\xi \longrightarrow \log\xi$ taking the positive reals to $\mathbb{R}$ (and $\xi \longrightarrow \log(-\xi)$ taking the negative reals to $\mathbb{R}$) which suggests that multiplication on the half-line in the Mellin setting corresponds to addition on the whole line in the Fourier setting, since $\log x + \log y = \log xy$. This crucial property becomes evident as we work out the following results (we only demonstrate results for $\mathcal{M}_{+}$ here; those for $\mathcal{M}_{-}$ are essentially identical):

Let

$$(\phi * \psi)(\xi) \equiv \int_{0}^{\infty} \phi(\frac{\xi}{\omega})\psi(\omega) \, \frac{d\omega}{\omega},$$

and observe

$$\mathcal{M}_{+} : \phi * \psi \longrightarrow \mathcal{M}_{+}\Big(\int_{0}^{\infty} \phi(\frac{\xi}{\omega})\psi(\omega) \, \frac{d\omega}{\omega}\Big) = \int_{0}^{\infty}\int_{0}^{\infty} \phi(\frac{\xi}{\omega})\psi(\omega) \, \frac{d\omega}{\omega}\xi^{2\pi i t-1/2} \, d\xi$$

$$= \int_{0}^{\infty} \psi(\omega)\int_{0}^{\infty} \phi(\xi)(\xi\omega)^{2\pi i t-1/2}\omega \, d\xi \, \frac{d\omega}{\omega}$$

$$= \Big(\int_{0}^{\infty} \phi(\xi)\xi^{2\pi i t-1/2} \, d\xi\Big)\Big(\int_{0}^{\infty} \psi(\omega)\omega^{2\pi i t-1/2} \, d\omega\Big).$$

Thus

$$\mathcal{M}_{+} : \phi * \psi \longrightarrow \mathcal{M}_{+}\phi(t)\mathcal{M}_{+}\psi(t). \qquad (1.22)$$

This establishes that the convolution property for the Mellin transform comes from dilating one of the convolved functions (compare to Eq. (1.5) for the Fourier transform, which comes from translating one of the convolved functions).

Now observe that for $r > 0$,

$$\mathcal{M}_+ : r^{1/2}\phi(r.) \longrightarrow \int_{-\infty}^{\infty} r^{1/2}\phi(r\xi)\xi_+^{2\pi it-1/2}\,d\xi = \int_0^\infty \phi(\xi)(\frac{\xi}{r})^{2\pi it-1/2}\,\frac{d\xi}{r^{1/2}}$$

$$= r^{-2\pi it}\int_0^\infty \phi(\xi)\xi^{2\pi it-1/2}\,d\xi.$$

Consequently,

$$\mathcal{M}_+ : r^{1/2}\phi(r.) \longrightarrow r^{-2\pi it}\mathcal{M}_+\phi(t) = e^{-2\pi i(\log r)t}\mathcal{M}_+\phi(t). \tag{1.23}$$

Also,

$$\mathcal{M}_+ : \xi^{2\pi i\tau}\phi(\xi) = e^{2\pi i(\log\xi)\tau}\phi(\xi) \longrightarrow \int_0^\infty \xi^{2\pi i\tau}\phi(\xi)\xi^{2\pi it-1/2}\,d\xi$$

$$= \int_0^\infty \phi(\xi)\xi^{2\pi i(t+\tau)-1/2}\,d\xi,$$

so

$$\mathcal{M}_+ : \xi^{2\pi i\tau}\phi(\xi) \longrightarrow \mathcal{M}_+\phi(t+\tau). \tag{1.24}$$

Thus, the Mellin transform sends dilation to modulation, and (logarithmic) modulation to translation (compare to Eqs. (1.6) and (1.7), where the Fourier transform sends translation to modulation and modulation to translation).

Finally, for $h > 0$,

$$\mathcal{M}_+ : \xi^{(h-1)/2}\phi(\xi^h) \longrightarrow \int_0^\infty \xi^{(h-1)/2}\phi(\xi^h)\xi^{2\pi it+1/2}\,\frac{d\xi}{\xi}$$

$$= \int_0^\infty \phi(\omega)\omega^{2\pi it/h}\omega^{1/2}\,\frac{d\omega}{h\omega} = \frac{1}{h}\int_0^\infty \phi(\omega)\omega^{2\pi it/h-1/2}\,d\omega,$$

and so

$$\mathcal{M}_+ : \xi^{(h-1)/2}\phi(\xi^h) \longrightarrow \frac{1}{h}\mathcal{M}_+\phi\Big(\frac{t}{h}\Big). \tag{1.25}$$

Thus we see that the Mellin transform acts upon exponentiation as the Fourier transform acts upon dilation (see Eq. (1.8)).

We conclude this section with a few more examples which will be useful in later discussion:

$$\mathcal{M}_+ : \xi^\alpha\phi(\xi) \longrightarrow \int_0^\infty \xi^\alpha\phi(\xi)\xi^{2\pi it-1/2}\,d\xi = \mathcal{M}_+\phi(t+\alpha/2\pi i), \tag{1.26}$$

$$\mathcal{M}_+ : e^{-|\xi|} \longrightarrow \int_0^\infty e^{-|\xi|}\xi^{2\pi it-1/2}\,d\xi = \Gamma(2\pi it+1/2), \tag{1.27}$$

and from Bateman [3], for $\mathrm{Re}\,\alpha > 0$,

$$\mathcal{M}_+ : e^{-\alpha\xi^2-\beta\xi} \longrightarrow \sqrt{2\alpha}^{-s}\Gamma(s)e^{\beta^2/8\alpha}D_{-s}[\beta/\sqrt{2\alpha}] \qquad (s = 2\pi it + 1/2), \tag{1.28}$$

where D_{-s} is the parabolic cylinder function

$$D_{-s}(x) = 2^{-s/2}e^{-x^2/4}\Psi(\frac{s}{2},\frac{1}{2};\frac{x^2}{2}),$$

and Ψ is the confluent hypergeometric series with integral representation

$$\Psi(a,c;x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1+t)^{c-a-1}\,dt.$$

We have seen thus far a brief introduction to Fourier and Mellin transforms. In particular, we notice that $\mathcal{F}$ acts on a function along all of $(-\infty,\infty)$, while $\mathcal{M}_+$ and $\mathcal{M}_-$ look only at $(0,\infty)$ and $(-\infty,0)$, respectively. In the next section, we shall characterize when exactly $\mathcal{F}$ takes a function on $(-\infty,\infty)$ to $(0,\infty)$ (or $(-\infty,0)$), in order that we might consider the *composition* of these two transformations.

2. THE PALEY-WIENER THEOREM

Define the Hardy space $H^2(\mathbb{C}_+)$ of functions f as those functions which are analytic in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, and whose L^2 norms along the lines $\mathrm{Im} z = y$ (constant) are bounded, i.e.,

$$\sup_{y>0}\left(\int_{-\infty}^{\infty} |f(x+iy)|^2\, dx\right)^{1/2} < \infty \qquad (x, y \in \mathbb{R}).$$

Similarly, define its companion space $H^2(\mathbb{C}_-)$ as those functions which are analytic in the lower half-plane $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im} z < 0\}$, and

$$\sup_{y<0}\left(\int_{-\infty}^{\infty} |f(x+iy)|^2\, dx\right)^{1/2} < \infty \qquad (x, y \in \mathbb{R}).$$

Functions $f = f(x)$ which may be analytically extended to $\mathbb{C}_+$ or $\mathbb{C}_-$ are referred to in signal analysis terminology as *analytic signals.* The Paley-Wiener theorem characterizes precisely when a function belongs to $H^2(\mathbb{C}_+)$ or $H^2(\mathbb{C}_-)$ (we prove the case for $H^2(\mathbb{C}_+)$ here)[4]:

(2.1) THEOREM (PALEY-WIENER). *$F \in H^2(\mathbb{C}_+)$ if and only if there exists $\phi \in L^2(\mathbb{R})$ with $\phi = 0$ on $(-\infty, 0)$ such that*

$$F(z) = \int_0^{\infty} \phi(\xi) e^{2\pi i \xi z}\, d\xi. \tag{2.1}$$

Let us precede the proof of the theorem with the following lemma:

(2.2) LEMMA. *Given any z_0 in $\mathbb{C}_+$,*

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{2\pi i \omega x}}{(\omega - z_0)}\, d\omega = \begin{cases} e^{2\pi i z_0 x} & (x > 0), \\ 0 & (x < 0). \end{cases} \tag{2.2}$$

Proof. (a) Suppose $x > 0$. Let $\Gamma_+ = \gamma_+ \cup [-R, R]$, where the curve γ_+ is given by $\gamma_+(\theta) = R(e^{i\theta}),\quad 0 < \theta < \pi$. Let $g(\omega) = e^{2\pi i \omega x}$. Then g is analytic on $\mathbb{C}$ and

$$\left|\frac{1}{2\pi i}\int_{\gamma_+} \frac{g(z)}{(z - z_0)}\, dz\right| = \left|\frac{1}{2\pi i}\int_{\gamma_+} \frac{e^{2\pi i z x}}{(z - z_0)}\, dz\right|$$

$$= \left|\frac{1}{2\pi i}\int_0^{\pi} \frac{e^{2\pi i (Re^{i\theta})x} i Re^{i\theta}}{(Re^{i\theta} - z_0)}\, d\theta\right| \le \frac{1}{2\pi}\int_0^{\pi}\left|\frac{e^{2\pi i (Re^{i\theta})x} i Re^{i\theta}}{(Re^{i\theta} - z_0)}\right| d\theta$$

$$\leq \frac{1}{2\pi}\int_0^\pi \frac{e^{-2\pi xR\sin\theta}R}{|Re^{i\theta}-z_0|}\,d\theta \leq \frac{1}{2\pi}\int_0^\pi \frac{e^{-2\pi xR\sin\theta}R}{R/2}\,d\theta \qquad (\text{for} \quad R\geq 2|z_0|)$$

$$= \frac{1}{\pi}\int_0^\pi e^{-2\pi xR\sin\theta}\,d\theta = \frac{2}{\pi}\int_0^{\pi/2} e^{-2\pi xR\sin\theta}\,d\theta.$$

Observe now that $\sin\theta \geq 2\theta/\pi$ on $[0,\pi/2]$, so

$$\frac{2}{\pi}\int_0^{\pi/2} e^{-2\pi xR\sin\theta}\,d\theta \leq \frac{2}{\pi}\int_0^{\pi/2} e^{-2\pi xR(2\theta/\pi)}\,d\theta = \frac{1-e^{2\pi xR}}{2\pi xR} \longrightarrow 0$$

as $R\longrightarrow\infty$. By the Cauchy integral formula, then

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{2\pi i\omega x}}{(\omega-z_0)}\,d\omega = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(\omega)}{(\omega-z_0)}\,d\omega$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\left(\int_{\gamma_+}\frac{g(z)}{(z-z_0)}\,dz + \int_{-R}^{R}\frac{g(z)}{(z-z_0)}\,dz\right) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma_+}\frac{g(z)}{(z-z_0)}\,dz$$

$$= \frac{1}{2\pi i}\int_{\Gamma_+}\frac{g(z)}{(z-z_0)}\,dz \qquad (\forall R>|z_0|)$$

$$= g(z_0) = e^{2\pi i z_0 x}.$$

(b) Suppose $x<0$. Let $\Gamma_- = \gamma_-\cup[-R,R]$ $\quad(R>0)$, where the curve γ_- is given by $\gamma_-(\theta) = Re^{-i\theta}$, $\quad 0<\theta<\pi$. Again, let $g(\omega)=e^{2\pi i\omega x}$. Then g is analytic on $\mathbb{C}$. Notice first that $z_0\in\mathbb{C}_+$ is always outside the curve Γ_- and g is analytic, so by Cauchy's theorem

$$\int_{\Gamma_-}\frac{g(z)}{(z-z_0)}\,dz = 0.$$

So,

$$\left|\frac{1}{2\pi i}\int_{\gamma_-}\frac{g(z)}{(z-z_0)}\,dz\right| = \left|\frac{1}{2\pi i}\int_{\gamma_-}\frac{e^{2\pi i z x}}{(z-z_0)}\,dz\right|$$

$$= \left|\frac{1}{2\pi i}\int_0^\pi \frac{e^{2\pi i(Re^{-i\theta})x}(-iRe^{-i\theta})}{(Re^{-i\theta}-z_0)}\,d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^\pi\left|\frac{e^{2\pi i(Re^{-i\theta})x}(-iRe^{-i\theta})}{(Re^{-i\theta}-z_0)}\right|d\theta \leq \frac{1}{2\pi}\int_0^\pi\frac{e^{2\pi xR\sin\theta}R}{|Re^{-i\theta}-z_0|}\,d\theta$$

$$\leq \frac{1}{2\pi}\int_0^\pi\frac{e^{-2\pi|x|R\sin\theta}R}{R/2}\,d\theta \qquad (\text{for} \quad R\geq 2|z_0|)$$

$$\longrightarrow 0 \quad \text{as} \quad R\longrightarrow\infty$$

by part (a). Therefore,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{2\pi i\omega x}}{(\omega-z_0)}\,d\omega = \lim_{R\to\infty}\frac{1}{2\pi i}\left(\int_{\Gamma_-}\frac{e^{2\pi i z x}}{(z-z_0)}\,dz - \int_{\gamma_-}\frac{e^{2\pi i z x}}{(z-z_0)}\,dz\right) = 0.$$

Hence,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{2\pi i\omega x}}{(\omega - z_0)}\,d\omega = \begin{cases} e^{2\pi i z_0 x} & (x>0),\\ 0 & (x<0); \end{cases}$$

proving the lemma. ∎

Proof of theorem. Suppose there exists $\phi \in L^2(\mathbb{R})$ with $\phi = 0$ on $(-\infty, 0)$ such that

$$F(z) = \int_0^\infty \phi(\xi)e^{2\pi i\xi z}\,d\xi = \int_0^\infty e^{-2\pi\xi y}\phi(\xi)e^{2\pi i\xi x}\,d\xi,$$

where $z = x + iy \in \mathbb{C}_+$. To see that F is analytic on $\mathbb{C}_+$, we shall show that

$$\lim_{z_0\to 0}\frac{F(z+z_0)-F(z)}{z_0}$$

exists for all $z \in \mathbb{C}_+$. So, choose any $z \in \mathbb{C}_+$ and observe that

$$\lim_{z_0\to 0}\frac{F(z+z_0)-F(z)}{z_0} = \lim_{z_0\to 0}\frac{\int_0^\infty \phi(\xi)e^{2\pi i\xi(z+z_0)}\,d\xi - \int_0^\infty \phi(\xi)e^{2\pi i\xi z}\,d\xi}{z_0}.$$

We can bring the limit inside the first integral by the dominated convergence theorem, since

$$\begin{aligned}\int_0^\infty |\phi(\xi)e^{2\pi i\xi(z+z_0)}|\,d\xi &\le \left(\int_0^\infty |\phi(\xi)|^2\,d\xi\right)^{1/2}\left(\int_0^\infty |e^{2\pi i\xi(z+z_0)}|^2\,d\xi\right)^{1/2}\\ &= \left(\int_0^\infty |\phi(\xi)|^2\,d\xi\right)^{1/2}\left(\int_0^\infty |e^{-4\pi\xi \mathrm{Im}(z+z_0)}|\,d\xi\right)^{1/2}\\ &< \infty \qquad for \quad |\mathrm{Im} z_0| < \mathrm{Im} z,\end{aligned}$$

since $\phi \in L^2(\mathbb{R})$. Thus,

$$\begin{aligned}\lim_{z_0\to 0}\frac{F(z+z_0)-F(z)}{z_0} &= \int_0^\infty \lim_{z_0\to 0}\left(\frac{e^{2\pi i\xi z_0}-1}{z_0}\right)\phi(\xi)e^{2\pi i\xi z}\,d\xi\\ &= \int_0^\infty 2\pi i\xi\phi(\xi)e^{2\pi i\xi z}\,d\xi.\end{aligned}$$

Hence, F is analytic on $\mathbb{C}_+$. On the other hand, by Plancherel's theorem,

$$\int_{-\infty}^\infty |F(x+iy)|^2\,dx = \int_0^\infty |e^{-2\pi\omega y}\phi(\omega)|^2\,d\omega \le \int_0^\infty |\phi(\omega)|^2\,d\omega.$$

Thus, since $\phi \in L^2(\mathbb{R})$

$$\sup_{y>0}\int_{-\infty}^\infty |F(x+iy)|^2\,dx \le \int_0^\infty |\phi(\omega)|^2\,d\omega \quad < \quad \infty.$$

So, $F \in H^2(\mathbb{C}_+)$.

Now suppose $F \in H^2(\mathbb{C}_+)$. Define F_b by

$$F_b(z) \equiv F(z+ib), \quad b>0, \quad z \in \mathbb{C}_+.$$

First we wish to show that F_b satisfies

$$F_b(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F_b(x)}{(x-z)}\,dx. \tag{2.3}$$

To do this, we will select a function k_α which approximates F_b, invoke Cauchy's integral formula for k_α, and then show that this carries over to F_b. Fix $b>0$, and define k_α by

$$k_\alpha(z) \equiv \frac{e^{i\alpha z}}{\alpha}\int_0^\alpha F_b(z+x)\,dx \tag{2.4}$$

for $\alpha > 0$. Notice here that since F_b is continuous, $\lim\limits_{\alpha\to 0} k_\alpha(z) = F_b(z)$. Let Γ_+ and γ_+ be as in the lemma above. By the Cauchy integral formula,

$$k_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_+} \frac{k_\alpha(u)}{(u-z)}\,du$$

for all R. For any $z \in \mathbb{C}_+$,

$$\left|\frac{1}{2\pi i}\int_{\gamma_+} \frac{k_\alpha(u)}{(u-z)}\,du\right| \le \frac{1}{2\pi}\int_0^\pi \left|\frac{k_\alpha(Re^{i\theta})iRe^{i\theta}}{(Re^{i\theta}-z)}\right| d\theta$$

$$\le \frac{1}{2\pi}\int_0^\pi \frac{|k_\alpha(Re^{i\theta})|R}{R/2}\,d\theta \qquad (\text{for} \quad R \ge 2|z|).$$

Consequently,

$$\left|\frac{1}{2\pi i}\int_{\gamma_+} \frac{k_\alpha(u)}{(u-z)}\,du\right| \le \frac{1}{\pi}\int_0^\pi |k_\alpha(Re^{i\theta})|\,d\theta.$$

Now observe that

$$|k_\alpha(Re^{i\theta})| \le \frac{e^{-\alpha R\sin\theta}}{\alpha}\int_0^\alpha |F_b(Re^{i\theta}+x)|\,dx$$

$$\le \frac{e^{-\alpha R\sin\theta}}{\alpha}\left(\int_0^\alpha dx\right)^{1/2}\left(\int_0^\alpha |F_b(Re^{i\theta}+x)|^2\,dx\right)^{1/2} \le \frac{Ce^{-\alpha R\sin\theta}}{\alpha^{1/2}},$$

where $C = \sup\limits_{b>0}\|F_b\|_2 < \infty$. Thus

$$\frac{1}{\pi}\int_0^\pi |k_\alpha(Re^{i\theta})|\,d\theta \le \frac{C}{\pi\alpha^{1/2}}\int_0^\pi e^{-\alpha R\sin\theta}\,d\theta \le \frac{2C}{\pi\alpha^{1/2}}\int_0^{\pi/2} e^{-\alpha R(2\theta/\pi)}\,d\theta$$

$$= \frac{C}{R\alpha^{3/2}}(1-e^{-\alpha R}) \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty.$$

As a result,

$$k_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_+}\frac{k_\alpha(u)}{(u-z)}\,du$$

$$= \lim_{R\to\infty}\left(\int_{\gamma_+}\frac{k_\alpha(u)}{(u-z)}\,du + \int_{-R}^{R}\frac{k_\alpha(x)}{(x-z)}\,dx\right)$$

$$= \int_{-\infty}^{\infty}\frac{k_\alpha(x)}{(x-z)}\,dx.$$

To see that this result goes over to the Cauchy integral formula for F_b, observe that

$$F_b(z) = \lim_{\alpha\to 0} k_\alpha(z) = \lim_{\alpha\to 0}\int_{-\infty}^{\infty}\frac{k_\alpha(x)}{x-z}\,dx$$

$$= \lim_{\alpha\to 0}\frac{1}{2\pi i}\left[\int_{|x|\le n}\frac{k_\alpha(x)}{x-z}\,dx + \int_{|x|>n}\frac{k_\alpha(x)}{x-z}\,dx\right] \qquad (\text{for} \quad all \quad n)$$

$$= \frac{1}{2\pi i}\int_{|x|\le n}\frac{F_b(x)}{x-z}\,dx + \lim_{\alpha\to 0}\frac{1}{2\pi i}\int_{|x|>n}\frac{k_\alpha(x)}{x-z}\,dx$$

since k_α is continuous on $[-n,n]$. To estimate the second term, note that

$$\int_{|x|>n}\left|\frac{k_\alpha(x)}{x-z}\right| dx \le \|k_\alpha\|_2\left(\int_{|x|>n}\frac{1}{|x-z|^2}\,dx\right)^{1/2} \le \frac{const.}{n}\|F_b\|_2.$$

But $\|F_b\|_2 < \infty$ for all $b > 0$. Thus

$$F_b(z) = \lim_{n\to\infty}\left(\frac{1}{2\pi i}\int_{|x|\le n}\frac{F_b(x)}{x-z}\,dx + \lim_{\alpha\to 0}\frac{1}{2\pi i}\int_{|x|>n}\frac{k_\alpha(x)}{x-z}\,dx\right)$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F_b(x)}{(x-z)}\,dx.$$

With this, the proof of the theorem is nearly complete. Observe that

$$F_b(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F_b(u)}{(u-z)}\,du = \int_{-\infty}^{\infty}\frac{1}{2\pi i(u-z)}\left(\int_{-\infty}^{\infty}\hat{F}_b(\xi)e^{2\pi i u\xi}\,d\xi\right)du$$

$$= \int_{-\infty}^{\infty}\hat{F}_b(\xi)\left(\int_{-\infty}^{\infty}\frac{e^{2\pi i u\xi}}{2\pi i(u-z)}\,du\right)d\xi = \int_0^{\infty}\hat{F}_b(\xi)e^{2\pi i\xi z}\,d\xi$$

by Lemma (2.2). Consequently, setting $z = x + ic \in \mathbb{C}_+$,

$$F_b(x+ic) = \int_0^{\infty}\hat{F}_b(\xi)e^{-2\pi\xi c}e^{2\pi i\xi x}\,d\xi.$$

Note now that for all $b, c > 0$,

$$\hat{F}_b(\xi)e^{-2\pi\xi c} = \int_{-\infty}^{\infty}F_b(t)e^{-2\pi\xi c}e^{-2\pi i t\xi}\,dt = \int_{-\infty}^{\infty}F_b(t)e^{-2\pi i(t-ic)\xi}\,dt$$

$$= \int_{-\infty}^{\infty} F_b(t+ic)e^{-2\pi it\xi}\,dt = \int_{-\infty}^{\infty} F_{b+c}(t)e^{-2\pi it\xi}\,dt = \hat{F}_{b+c}(\xi).$$

Finally, set

$$\phi(\xi) = \begin{cases} e^{2\pi\epsilon\xi}\hat{F}_\epsilon(\xi) & (\xi \geq 0), \\ 0 & (\xi < 0). \end{cases}$$

The previous result ensures that ϕ is independent of $\epsilon > 0$. So, for $\mathrm{Re}z > \epsilon$,

$$\int_0^\infty \phi(\xi)e^{2\pi i\xi z}\,d\xi = \int_0^\infty \hat{F}_\epsilon(\xi)e^{2\pi\xi\epsilon}e^{2\pi i\xi z}\,d\xi = \int_0^\infty \hat{F}_\epsilon(\xi)e^{2\pi i\xi(z-i\epsilon)}\,d\xi$$

$$= F_\epsilon(z-i\epsilon) = F(z).$$

To see that $\phi \in L^2(\mathbb{R})$, we note that

$$\int_{-\infty}^{\infty} |\phi(\xi)|^2\,d\xi = \lim_{b\to 0}\int_0^\infty e^{-4\pi\xi b}|\phi(\xi)|^2\,d\xi$$

$$= \lim_{b\to 0}\int_0^\infty |e^{-2\pi\xi b}\phi(\xi)|^2\,d\xi = \lim_{b\to 0}\int_0^\infty |\hat{F}_b(\xi)|^2\,d\xi$$

$$= \lim_{b\to 0}\int_{-\infty}^{\infty} |F_b(x)|^2\,dx \qquad (\text{by} \quad Plancherel)$$

$$< \infty \quad since \quad F \in H^2(\mathbb{C}_+).$$

Hence, we have determined ϕ as desired, and the proof is complete. ∎

By the same method of proof, the following analogous result holds:

(2.5) **Theorem**. *$F \in H^2(\mathbb{C}_-)$ if and only if there exists $\phi \in L^2(\mathbb{R})$ with $\phi = 0$ on $(0,\infty)$ such that*

$$F(z) = \int_{-\infty}^{0} \phi(\xi)e^{2\pi i\xi z}\,d\xi. \tag{2.5}$$

From Eqs. (2.1) and (2.5) it follows immediately that

$$L^2(\mathbb{R}) = H^2_+(\mathbb{R}) \oplus H^2_-(\mathbb{R}), \tag{2.6}$$

where $H^2_\pm(\mathbb{R}) = \{f : f = \lim_{y\to 0} F(x \pm iy), \quad \exists F \in H^2(\mathbb{C}_\pm)\}$, and that

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |f_+(x)|^2\,dx + \int_{-\infty}^{\infty} |f_-(x)|^2\,dx, \tag{2.7}$$

where

$$f(x) = f_+(x) + f_-(x) = \int_0^\infty \hat{f}(\xi)e^{2\pi ix\xi}\,d\xi + \int_{-\infty}^{0} \hat{f}(\xi)e^{2\pi ix\xi}\,d\xi. \tag{2.8}$$

In the same vein, given any function $f \in L^2(\mathbb{R})$ we are able to construct from it a function F in $H^2(\mathbb{C}_+)$ or $H^2(\mathbb{C}_-)$ by means of the *Hilbert transform*,

$$(\mathcal{H}f)(x) = \mathrm{PV}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy. \tag{2.9}$$

This integral is a Cauchy principal value, by which we mean

$$\mathrm{PV}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy.$$

We proceed as follows: Define g_ϵ by

$$g_\epsilon(y) \equiv \begin{cases} y^{-1} & |y| > \epsilon, \\ 0 & \text{elsewhere,} \end{cases} \tag{2.10}$$

and note that

$$\begin{aligned} \lim_{\epsilon\to 0}\hat{g}_\epsilon(\xi) &= \lim_{\epsilon\to 0}\int_{|y|>\epsilon} \frac{e^{-2\pi i y\xi}}{y}\, dy \\ = -2i\int_0^{\infty}\frac{\sin 2\pi\xi y}{y}\, dy &= \begin{cases} -2i(\frac{\pi}{2}) & (\xi > 0) \\ -2i(\frac{-\pi}{2}) & (\xi < 0) \end{cases} \\ &= -i\pi sgn(\xi) \qquad (\xi \neq 0). \end{aligned}$$

So,

$$\begin{aligned} (\mathcal{H}f)(x) &= \mathrm{PV}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-y|>\epsilon}\frac{f(y)}{x-y}\, dy \\ &= \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|y|>\epsilon}\frac{f(x-y)}{y}\, dy = \lim_{\epsilon\to 0}\frac{1}{\pi}(f * g_\epsilon)(x) \\ &= \lim_{\epsilon\to 0}\frac{1}{\pi}(\hat{f}(\xi)\hat{g}_\epsilon(\xi))^\vee(x) = \frac{1}{\pi}(\hat{f}(\xi)(-i\pi)sgn(\xi))^\vee(x), \end{aligned}$$

and thus

$$(\mathcal{H}f)(x) = -i(\hat{f}(\xi)sgn(\xi))^\vee(x). \tag{2.11}$$

We see then by letting

$$p = \frac{f + i\mathcal{H}f}{2} \quad , \quad q = \frac{f - i\mathcal{H}f}{2} \tag{2.12}$$

that $p, q \in L^2(\mathbb{R})$ and

$$\begin{aligned} F_+(z) \equiv \int_{-\infty}^{\infty} \hat{p}(\xi)e^{2\pi i\xi z}\, d\xi &= \int_{-\infty}^{\infty}\frac{\hat{f}(\xi) + \hat{f}(\xi)sgn(\xi)}{2}e^{2\pi i\xi z}\, d\xi \\ &= \int_0^{\infty}\hat{f}(\xi)e^{2\pi i\xi z}\, d\xi \quad \in H^2(\mathbb{C}_+), \end{aligned}$$

while

$$F_-(z) \equiv \int_{-\infty}^{\infty} \hat{q}(\xi) e^{2\pi i \xi z}\, d\xi = \int_{-\infty}^{\infty} \frac{\hat{f}(\xi) - \hat{f}(\xi) sgn(\xi)}{2} e^{2\pi i \xi z}\, d\xi$$

$$= \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i \xi z}\, d\xi \quad \in H^2(\mathbb{C}_-).$$

Up to this point, we have concentrated on the L^2 theory of the Fourier transform, utilizing the fact that $\mathcal{F}$ maps $L^2(\mathbb{R})$ unitarily onto $L^2(\mathbb{R})$. Before moving on, we make some remarks about L^1 theory: Let p and q be conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$) such that $1 \leq p < 2$. From the Hausdorff-Young inequality

$$\|\hat{f}\|_q \leq \|f\|_p$$

one concludes that

$$\mathcal{F} : L^p(\mathbb{R}) \longrightarrow L^q(\mathbb{R}).$$

For the case $p = 1$, the stronger result

$$\mathcal{F} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$$

holds, where $C_0(\mathbb{R})$ is the space of continuous bounded functions on $\mathbb{R}$.

We now define the space $H^1(\mathbb{C}_+)$ in the same fashion as we did $H^2(\mathbb{C}_+)$: Let $H^1(\mathbb{C}_+)$ be the space of all functions F which are analytic in the upper half-plane $\mathbb{C}_+$, and whose L^1 norms along the lines Im$z = y$ (constant) are bounded, i.e.,

$$\sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|\, dx < \infty \qquad (x, y \in \mathbb{R}).$$

A function $f \in L^1(\mathbb{R})$ is said to be an *analytic L^1-signal* if f extends analytically to a function $F \in H^1(\mathbb{C}_+)$. We define the H^1-norm of an analytic L^1-signal to be

$$\|f\|_{H^1} = \sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|\, dx,$$

where F is the unique analytic extension of f to $H^1(\mathbb{C}_+)$. With these definitions we have the following theorem, similar in content and proof to the more difficult half of the Paley-Wiener theorem:

(2.13) THEOREM. *If f is an analytic L^1-signal, then*

$$\hat{f} = 0 on (-\infty, 0]. \tag{2.13}$$

We also state the following results, due to Hardy, without proof:

(2.14) **Theorem.** *If f is an analytic L^1-signal, then*

$$\int_0^\infty |\hat{f}(\xi)| \frac{d\xi}{\xi} \leq C_1 \|f\|_{H^1} \tag{2.14)(i}$$

and

$$\left(\int_0^\infty |\hat{f}(\xi)|^2 \frac{d\xi}{\xi}\right)^{1/2} \leq C_2 \|f\|_{H^1} \tag{2.14)(ii}$$

where C_1 and C_2 are constants independent of f.

These results will be necessary in our discussion of the classical Mellin transform and convolution in the next section.

So, via the Paley-Wiener theorem and the Hilbert transform, we see what is necessary to restrict the support of a function's Fourier transform to the half-line. The following section builds upon this notion as we move on to consider the composition of the Fourier and Mellin transforms.

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3. THE K-TRANSFORM

(3.1) Definition. *The $\mathcal{K}$-transform is the composition*

$$\mathcal{K} = (\mathcal{M}_+ \oplus \mathcal{M}_-) \circ \mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \tag{3.1}$$

of the Fourier and Mellin transforms.

As $\mathcal{K}$ is the composition of unitary mappings, it too will be unitary and it will take $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. There is an explicit expression for $\mathcal{K}$ as integral operator. Set

$$K(x,t) = (2\pi i x)^{-(2\pi i t+1/2)}\Gamma(2\pi i t + 1/2). \tag{3.2}$$

(3.3) Theorem. *On $L^2(\mathbb{R})$ the $\mathcal{K}$-transform is given by*

$$F_+(t) = (\mathcal{M}_+\hat{f})(t) = \int_{-\infty}^{\infty} f(x)K(x,t)\,dx, \qquad (3.3)(i)$$

and

$$F_-(t) = (\mathcal{M}_-\hat{f})(t) = \int_{-\infty}^{\infty} f(x)K(-x,t)\,dx; \qquad (3.3)(ii)$$

furthermore, on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ the inverse $\mathcal{K}$-transform is given by

$$f(x) = \int_{-\infty}^{\infty} F_+(t)\overline{K(x,t)}\,dt + \int_{-\infty}^{\infty} F_-(t)\overline{K(-x,t)}\,dt \qquad (3.3)(iii)$$

where in all cases $K(x,t)$ is defined by (3.2).

Proof. In the right half-plane $\{z \in \mathbb{C} : \mathrm{Re} z > 0\}$ the Γ-function is defined by [5]

$$\Gamma(z) = s^z \int_0^{e^{i\delta}\infty} e^{-su}u^z \frac{du}{u}, \tag{3.4}$$

where $-\pi/2 < Arg(s) + \delta < \pi/2$, or $Arg(s) + \delta = \pm\pi/2$ and $0 < \mathrm{Re} z < 1$. We will consider only the case where $\delta = 0$. Now by definition,

$$\begin{aligned} F_+(t) = (\mathcal{M}_+\hat{f})(t) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx \right) \xi_+^{2\pi i t - 1/2}\,d\xi \\ &= \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} e^{-2\pi i x \xi} \xi_+^{2\pi i t+1/2} \frac{d\xi}{\xi} \right) dx. \end{aligned}$$

The inner integral can be evaluated using the definition given above for the Γ-function, setting $s = 2\pi i x, u = \xi$, and $z = 2\pi i t + 1/2$ (and $\delta = 0$, remember). For these choices

$$\Gamma(2\pi i t + 1/2) = (2\pi i x)^{2\pi i t + 1/2} \int_0^\infty e^{-2\pi i x \xi} \xi^{2\pi i t + 1/2} \frac{d\xi}{\xi},$$

so

$$K(x,t) = (2\pi i x)^{-(2\pi i t + 1/2)} \Gamma(2\pi i t + 1/2) = \int_0^\infty e^{-2\pi i x \xi} \xi^{2\pi i t + 1/2} \frac{d\xi}{\xi}.$$

Thus

$$F_+(t) = \int_{-\infty}^\infty f(x)(2\pi i x)^{-(2\pi i t + 1/2)} \Gamma(2\pi i t + 1/2)\, dx.$$

Similarly,

$$\begin{aligned} F_-(t) = (\mathcal{M}_- \hat{f})(t) &= \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(x) e^{-2\pi i x \xi}\, dx \right) \xi_-^{2\pi i t - 1/2}\, d\xi \\ &= \int_{-\infty}^\infty f(x) \left(\int_{-\infty}^0 e^{-2\pi i x \xi} (-\xi)^{2\pi i t - 1/2}\, d\xi \right) dx \\ &= \int_{-\infty}^\infty f(x) \left(\int_0^\infty e^{2\pi i x \xi} \xi^{2\pi i t + 1/2} \frac{d\xi}{\xi} \right) dx. \end{aligned}$$

Thus

$$F_-(t) = \int_{-\infty}^\infty f(x) K(-x,t)\, dx.$$

To determine the inverse transforms, observe that

$$\overline{K(x,t)} = \overline{\int_0^\infty e^{-2\pi i x \xi} \xi^{2\pi i t + 1/2} \frac{d\xi}{\xi}} = \int_0^\infty e^{2\pi i x \xi} \xi^{-2\pi i t + 1/2} \frac{d\xi}{\xi} = K(-x,-t).$$

Thus

$$\begin{aligned} f_+(x) = \int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi &= \mathcal{F}^{-1} \circ \mathcal{M}_+^{-1} \circ F_+(x) \\ &= \int_0^\infty \left(\int_{-\infty}^\infty F_+(t) \xi_+^{-2\pi i t - 1/2}\, dt \right) e^{2\pi i x \xi}\, d\xi \\ &= \int_{-\infty}^\infty F_+(t) \left(\int_0^\infty e^{2\pi i x \xi} \xi^{-2\pi i t + 1/2} \frac{d\xi}{\xi} \right) dt. \end{aligned}$$

Consequently,

$$f_+(x) = \int_{-\infty}^\infty F_+(t) K(-x,-t)\, dt = \int_{-\infty}^\infty F_+(t) \overline{K(x,t)}\, dt.$$

Similarly,

$$\begin{aligned} f_-(x) = \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi &= \mathcal{F}^{-1} \circ \mathcal{M}_-^{-1} \circ F_-(x) \\ &= \int_{-\infty}^0 \left(\int_{-\infty}^\infty F_-(t) \xi_-^{-2\pi i t - 1/2}\, dt \right) e^{2\pi i x \xi}\, d\xi \end{aligned}$$

$$= \int_{-\infty}^{\infty} F_-(t)\Big(\int_{-\infty}^{0} e^{2\pi i x\xi}(-\xi)^{-2\pi i t-1/2}\, d\xi\Big)\, dt$$

$$= \int_{-\infty}^{\infty} F_-(t)\Big(\int_{0}^{\infty} e^{-2\pi i x\xi}\xi^{-2\pi i t+1/2}\,\frac{d\xi}{\xi}\Big)\, dt.$$

Consequently,

$$f_-(x) = \int_{-\infty}^{\infty} F_-(t)K(x,-t)\, dt = \int_{-\infty}^{\infty} F_-(t)\overline{K(-x,t)}\, dt\ .$$

This completes the proof. ∎

Using the asymptotic estimate

$$\Gamma(2\pi it + 1/2) \simeq \sqrt{2\pi}e^{2\pi it\mathrm{Log}(2\pi it+1/2)-2\pi it-1/2}\Big(1 + \frac{1}{12(2\pi it+1/2)}\Big)$$

for the Γ-function, we can derive a corresponding asymptotic estimate for the kernel $K = K(x,t)$:

$$K(x,t) = (2\pi ix)^{-(2\pi it+1/2)}\Gamma(2\pi it + 1/2)$$

$$\simeq \sqrt{2\pi}e^{-(2\pi it+1/2)\mathrm{Log}(2\pi ix)}e^{2\pi it\mathrm{Log}(2\pi it+1/2)-2\pi it-1/2}\Big(1 + \frac{1}{12(2\pi it+1/2)}\Big)\ .$$

But for z in $\mathbb{C}$,

$$\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z),$$

and so

$$e^{2\pi it\mathrm{Log}(2\pi it+1/2)} = e^{2\pi it(\log|2\pi it+1/2|+i\mathrm{Arg}(2\pi it+1/2))}$$

$$= |2\pi it + 1/2|^{2\pi it}e^{-2\pi t\mathrm{Arg}(2\pi it+1/2)}$$

while

$$e^{-(2\pi it+1/2)\mathrm{Log}(2\pi ix)} = e^{-(2\pi it+1/2)\log|2\pi x|}e^{-(2\pi it+1/2)i\,\mathrm{Arg}(2\pi ix)}$$

$$= \Big(\frac{1}{2\pi|x|}\Big)^{2\pi it+1/2}e^{\pi^2 t\,sgn(x)}e^{-i\pi\,sgn(x)/4}.$$

Consequently,

$$K(x,t) \simeq \sqrt{2\pi}\Big(\frac{1}{2\pi e|x|}\Big)^{2\pi it+1/2}|2\pi it+1/2|^{2\pi it}e^{-i\pi\,sgn(x)/4} \tag{3.5}$$

$$\times\quad e^{\pi t(\pi\,sgn(x)-2\,\mathrm{Arg}(2\pi it+1/2))}\Big(1 + \frac{1}{12(2\pi it+1/2)}\Big).$$

Thus the asymptotic behavior of K depends critically on the sign of both x and t because these will determine the boundedness of the exponential term. First, suppose that $x > 0$. Then

$$\lim_{t\to\infty} \pi t(\pi\,sgn(x) - 2\,\mathrm{Arg}(2\pi it + 1/2)) = \lim_{t\to\infty} \pi t(\pi - 2\,\mathrm{Arg}(2\pi it + 1/2))$$

$$= \lim_{t\to\infty} \frac{\pi - 2\arctan(4\pi t)}{1/\pi t} = \lim_{t\to\infty} \frac{8\pi/(1+(4\pi t)^2)}{1/\pi t^2}$$

$$= \lim_{t\to\infty} \frac{8\pi^2 t^2}{(1+16\pi^2 t^2)} = 1/2,$$

while

$$\lim_{t\to-\infty} \pi t(\pi\, sgn(x) - 2\,\mathrm{A}rg(2\pi i t + 1/2)) = \lim_{t\to-\infty} \pi t(\pi - 2\,\mathrm{A}rg(2\pi i t + 1/2))$$

$$= \lim_{t\to\infty} -\pi t(\pi + 2\,\mathrm{A}rg(2\pi i t + 1/2)) = -\infty,$$

since $\mathrm{A}rg(2\pi i t + 1/2) > 0$ for all $t > 0$. Now suppose that $x < 0$. In the same fashion as above, we see

$$\lim_{t\to\infty} \pi t(\pi\, sgn(x) - 2\,\mathrm{A}rg(2\pi i t + 1/2)) = \lim_{t\to\infty} \pi t(-\pi - 2\,\mathrm{A}rg(2\pi i t + 1/2))$$

$$= \lim_{t\to\infty} -\pi t(\pi + 2\,\mathrm{A}rg(2\pi i t + 1/2)) = -\infty,$$

while

$$\lim_{t\to-\infty} \pi t(\pi\, sgn(x) - 2\,\mathrm{A}rg(2\pi i t + 1/2)) = \lim_{t\to-\infty} \pi t(-\pi - 2\,\mathrm{A}rg(2\pi i t + 1/2))$$

$$= \lim_{t\to\infty} -\pi t(-\pi + 2\,\mathrm{A}rg(2\pi i t + 1/2)) = \lim_{t\to\infty} \pi t(\pi - 2\,\mathrm{A}rg(2\pi i t + 1/2)) = 1/2.$$

In particular, therefore, the term

$$e^{\pi t(\pi\, sgn(x) - 2\,\mathrm{A}rg(2\pi i t + 1/2))}$$

in the asymptotic estimate for K is a bounded function of x and t and so K itself is uniformly bounded in both x and t.

As the $\mathcal{K}$-transform is less familiar than its individual components, some examples will perhaps make its fundamental properties clearer.

(3.6) EXAMPLES.

(i) The $\mathcal{K}$-transform sends dilation to modulation; that is,

$$\mathcal{M}_\pm \circ \mathcal{F} : a^{1/2} f(ax) \longrightarrow a^{2\pi i t}(\mathcal{M}_\pm \hat{f})(t) = a^{2\pi i t} F_\pm(t). \qquad (3.6)(i)$$

This result follows directly from Eqs. (1.8) and (1.23), by which

$$\mathcal{M}_+ \circ \mathcal{F} : a^{1/2} f(ax) \longrightarrow \mathcal{M}_+\Big[\frac{1}{a^{1/2}} \hat{f}\Big(\frac{\xi}{a}\Big)\Big]$$

$$= a^{2\pi it}(\mathcal{M}_+\hat{f})(t) = a^{2\pi it}F_+(t).$$

The case for $\mathcal{M}_- \circ \mathcal{F}$ is identical.

(ii) For the *sinc*-function,

$$\mathcal{M}_\pm \circ \mathcal{F} : \frac{\sin 2\pi x}{\pi x} \longrightarrow \frac{1}{2\pi it + 1/2}. \tag{3.6)(ii}$$

This result follows by direct calculation. Letting χ be the characteristic function on the interval [-1,1], we obtain

$$\mathcal{M}_+ \circ \mathcal{F} : \frac{\sin 2\pi x}{\pi x} \longrightarrow \mathcal{M}_+[\chi(\xi)]$$

$$= \int_{-\infty}^{\infty} \chi(\xi)\xi_+^{2\pi it-1/2}\, d\xi = \int_0^1 \xi^{2\pi it-1/2}\, d\xi$$

$$= \frac{\xi^{2\pi it+1/2}}{2\pi it+1/2}\bigg|_0^1 = \frac{1}{2\pi it+1/2}.$$

Again, $\mathcal{M}_- \circ \mathcal{F}$ follows identically.

(iii) For the Poisson kernel,

$$\mathcal{M}_\pm \circ \mathcal{F} : \frac{1}{\pi}\left(\frac{1}{1+x^2}\right) \longrightarrow \Gamma(2\pi it + 1/2). \tag{3.6)(iii}$$

Here we reference Eqs (1.12)(ii) and (1.27), by which

$$\mathcal{M}_+ \circ \mathcal{F} : \frac{1}{\pi}\left(\frac{1}{1+x^2}\right) \longrightarrow \mathcal{M}_+[e^{-|\xi|}] = \Gamma(2\pi it + 1/2).$$

Once more, the same follows for $\mathcal{M}_- \circ \mathcal{F}$.

(iv) For the normalized Gaussian with mean b and variance a,

$$\mathcal{M}_+ \circ \mathcal{F} : \frac{1}{a\sqrt{2\pi}}e^{-(x-b)^2/2a^2} \longrightarrow \mathcal{M}_+\left[\frac{e^{-2\pi ib\xi}}{a\sqrt{2\pi}}\mathcal{F}(e^{-x^2/2a^2})\right] \qquad (\text{by } Eq. \quad (1.6))$$

$$= \mathcal{M}_+\left[\frac{e^{-2\pi ib\xi}}{a\sqrt{2\pi}}\mathcal{F}(e^{-\pi(x/a\sqrt{2\pi})^2})\right] = \mathcal{M}_+[e^{-2\pi ib\xi}e^{-\pi(a\sqrt{2\pi}\xi)^2}] \qquad (\text{by } Eq. \quad (1.8))$$

$$= \mathcal{M}_+[e^{-\frac{1}{2}(2a\pi)^2\xi^2-2\pi ib\xi}].$$

Now we substitute directly into Eq. (1.28) with $\alpha = \frac{1}{2}(2\pi a)^2, \beta = 2\pi ib$, and $s = 2\pi it + 1/2$ to obtain

$$\mathcal{M}_+[e^{-\frac{1}{2}(2\pi a)^2\xi^2-2\pi ib\xi}] = (2\pi a)^{-s}\Gamma(s)e^{(ib/2a)^2}2^{-s/2}e^{-(ib/2a)^2}\Psi(\frac{s}{2}, \frac{1}{2}; \frac{-b^2}{2a^2})$$

$$= 2^{-s/2}(2\pi a)^{-s}\Gamma(s)\Psi(\frac{s}{2},\frac{1}{2};\frac{-b^2}{2a^2}).$$

Note that we may rewrite this result as

$$F_+(t) = 2^{-(\pi it+1/4)}K(-ia,t)\Psi(\pi it+1/4,\frac{1}{2};\frac{-b^2}{2a^2}).$$

To determine $\mathcal{M}_- \circ \mathcal{F}$, we utilize the identity $\mathcal{M}_+[\phi(-\xi)] = \mathcal{M}_-\phi$, so substituting $\xi' = -\xi$ above yields

$$F_-(t) = \mathcal{M}_+[e^{-\frac{1}{2}(2a\pi)^2(\xi')^2+2\pi ib\xi'}] = 2^{-s/2}(2\pi a)^{-s}\Gamma(s)\Psi(\frac{s}{2},\frac{1}{2};\frac{-b^2}{2a^2}) = F_+(t).$$

In Eq. (3.6)(i) we observe the critical property that the $\mathcal{K}$-transform sends *scaling* to *modulation.* Let's investigate the action of $\mathcal{K}$ upon *translation* (regarding now only the case of $\mathcal{M}_+ \circ \mathcal{F}$):

$$\mathcal{M}_+ \circ \mathcal{F} : f(v+x) \longrightarrow \mathcal{M}_+(e^{2\pi ix\xi}\hat{f}) = \int_0^\infty e^{2\pi ix\xi}\hat{f}(\xi)\xi^{2\pi it-1/2}\,d\xi.$$

Representing $\hat{f}$ by the inverse Mellin transform of F_+ (where $F_+ = (\mathcal{M}_+ \circ \mathcal{F})f$), this expression becomes

$$(\mathcal{M}_+ \circ \mathcal{F}) : f(v+x) \longrightarrow \int_0^\infty e^{2\pi ix\xi}\Big(\int_{-\infty}^\infty F_+(s)\xi^{-2\pi is-1/2}\,ds\Big)\xi^{2\pi it-1/2}\,d\xi.$$

If Fubini's theorem is applicable here, one obtains the result

$$(\mathcal{M}_+ \circ \mathcal{F}) : f(v+x) \longrightarrow \int_{-\infty}^\infty F_+(s)\Big(\int_0^\infty e^{2\pi ix\xi}\xi^{2\pi i(t-s)-1}\,d\xi\Big)\,ds.$$

Without further restrictions on F_+, the inner integral is singular and so only converges for $0 < \mathrm{Re}(2\pi i(t-s)) < 1$ (see Eq. 3.4), which is not the case here since $s,t \in \mathbb{R}$. But if, for instance, $F_+ = \mathcal{K}f$ where f is an analytic L^1-signal, then $\hat{f}(0) = 0$ and the singularity of the inner integral may be compensated for. If we interpret this integral formally as a gamma function, we then have a kernel $\mathcal{T}_x$ given by

$$\mathcal{T}_x(s) = (-2\pi ix)^{-2\pi is}\Gamma(2\pi is)$$

such that

$$(\mathcal{M}_+ \circ \mathcal{F}) : f(v+x) \longrightarrow \int_{-\infty}^\infty F_+(s)\mathcal{T}_x(t-s)\,ds = \mathcal{T}_x * F_+. \tag{3.7}$$

We observe that the term $\xi^{2\pi i(t-s)-1}$ is suggestive of the kernel of the classical Mellin transform

$$\mathfrak{M}\phi(t) \equiv \mathcal{M}\phi(-2\pi it) = \int_0^\infty \phi(\xi)\xi^{-2\pi it-1}\,d\xi \tag{3.8}$$

with inversion formula

$$\phi(\xi) = \int_{-\infty}^{\infty} \mathfrak{M}\phi(t)\xi^{2\pi i t}\, dt. \tag{3.9}$$

Notice that $\mathfrak{M}\phi$ exists if

$$\int_0^{\infty} |\phi(\xi)| \frac{d\xi}{\xi} < \infty,$$

which is precisely Eq. (2.14)(i) when $\phi = \hat{f}$ and f is an L^1-analytic signal. This classical Mellin transform turns out to be a nice setting in which to look at *convolution* with respect to the $\mathcal{K}$-transform. If f and g are L^1-analytic, then their convolution is also L^1-analytic. Defining $\mathfrak{K} = \mathfrak{M} \circ \mathcal{F}$, we obtain the interesting result

$$\begin{aligned}
\mathfrak{K}(f * g) = \mathfrak{M} \circ \mathcal{F}(f * g) = \mathfrak{M}(\hat{f}\hat{g}) \\
= \mathfrak{M}\Big(\int_{-\infty}^{\infty} \mathfrak{M}\hat{f}(u)\xi^{2\pi i u}\, du \int_{-\infty}^{\infty} \mathfrak{M}\hat{g}(v)\xi^{2\pi i v}\, dv\Big) \\
= \mathfrak{M}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathfrak{M}\hat{f}(u)\mathfrak{M}\hat{g}(v)\xi^{2\pi i(u+v)}\, dudv\Big) \\
= \mathfrak{M}\Big(\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} \mathfrak{M}\hat{f}(u)\mathfrak{M}\hat{g}(v-u)\, du\Big)\xi^{2\pi i v}\, dv\Big) \\
= \mathfrak{M} \circ \mathfrak{M}^{-1} \circ (\mathfrak{M}\hat{f} * \mathfrak{M}\hat{g});
\end{aligned}$$

consequently,

$$\mathfrak{K} : f * g \longrightarrow \mathfrak{K}f * \mathfrak{K}g. \tag{3.10}$$

In other words, $\mathfrak{K}$ sends *convolution* to *convolution.* By the same argument, one deduces the similar result for $\mathcal{K}$:

$$\mathcal{K}(f * g)(t) = (\mathcal{K}f * \mathcal{K}g)(t - \tfrac{1}{4\pi i}). \tag{3.11}$$

This framework with convolution and the $\mathfrak{K}$-transform is tied in directly with Altes' pursuit of a signal representation which is *independent of time shift and scale change.*[6] Let f and g be analytic signals such that $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Recall that $f^*(x) = \overline{f(-x)}$. By the Paley-Wiener theorem, if a function $h = h(x)$ is in $H^2_{\pm}(\mathbb{R})$, then $h^* \in H^2_{\pm}(\mathbb{R})$ since conjugation does not effect $\mathcal{F}h$ supported only on $(0, \infty)$ or $(-\infty, 0)$. Observe that

$$\mathfrak{K} : f^* \longrightarrow \mathfrak{M}\Big(\int_0^{\infty} \overline{\hat{f}(\xi)}\xi^{-2\pi i t - 1}\, d\xi\Big) = \overline{\int_0^{\infty} \hat{f}(\xi)\xi^{2\pi i t - 1}\, d\xi},$$

thus

$$\mathfrak{K} : f^* \longrightarrow (\mathfrak{K}f)^*. \tag{3.12}$$

From Eq. (3.10), then,

$$\mathfrak{K} : f * g^* \longrightarrow \mathfrak{K}f * (\mathfrak{K}g)^*. \tag{3.13}$$

Altes presents the "$|\mathcal{F}|^2 - |\mathfrak{M}|^2$ transform" (denoted here by $\mathfrak{G}$) to be the magnitude-squared (classical) Mellin transform of a signal's energy density spectrum [6]; that is

$$\mathfrak{G}f = \left|\mathfrak{M}\left(|\mathcal{F}f|^2\right)\right|^2 = \left|\int_0^\infty |\hat{f}(\xi)|^2 \xi^{-2\pi i t-1}\, d\xi\right|^2.$$

This idea arises in the context of work to model mammalian hearing. The purpose of such a transform is to "wipe out" time shift and scale change in a signal, utilizing the properties that $|\mathcal{F}f|$ is invariant under translation of f and $|\mathfrak{M}\phi|$ is invariant under dilation of ϕ. We have in fact captured this transform in a more general setting with the $\mathfrak{K}$-transform, since

$$\left|\mathfrak{M}\left(\mathcal{F}f\overline{\mathcal{F}g}\right)\right|^2 = |\mathfrak{M} \circ \mathcal{F}(f * g^*)|^2 = |\mathfrak{K}(f * g^*)|^2 = |\mathfrak{K}f * (\mathfrak{K}g)^*|^2.$$

(Altes' transform is the special case $f = g$.)

We note here that Altes, in the same paper, presents one other slightly more complicated transform involving composition of the Fourier and Mellin transforms (the "$\hat{\mathcal{F}} - \hat{\mathfrak{M}}$ transform"), which preserves some phase information (unlike the $|\mathcal{F}|^2 - |\mathfrak{M}|^2$ transform). It may be of interest to see if this second transformation may be interpreted in the context of the $\mathcal{K}$ and $\mathfrak{K}$ transforms. We leave this question as an open problem at present. Now our interests turn to the wavelet transform and the Wigner-Ville distribution, as we attempt in the next section to incorporate these two concepts with the $\mathcal{K}$-transform.

4. WAVELETS AND WIGNER-VILLE

We have so far seen an introduction to the Fourier and Mellin tranforms, some basic properties of each, and the presentation of the $\mathcal{K}$-transform as a composition of these first two. Now let's investigate how this all might be tied in with wavelet analysis and the Wigner-Ville distribution. First, recall that the **Gabor transform** (or **short-time Fourier transform**) is defined by the mapping

$$f \longrightarrow S_g f(t,\xi) = \int_{-\infty}^{\infty} f(x)\overline{g(x-t)}e^{-2\pi i x\xi}\,dx, \tag{4.1}$$

sending the signal f into a two-dimensional function living in the time-frequency plane (t,ξ).[7] The basic idea of the Gabor transform is that if you want to know the particular time at which particular frequencies exist in a signal, then look at a small piece of the signal around the desired time t and take its Fourier transform.[8] Many properties of the Fourier transform carry over to the Gabor transform; however, the analysis here depends critically on the choice of the window g (when Gabor introduced this transform, the special case he considered was a Gaussian window).[7] Given a window function $g(t)$, define its bandwidth $\Delta\xi$ by

$$\Delta\xi = \frac{\left(\int_{-\infty}^{\infty} \xi^2 |\hat{g}(\xi)|^2\,d\xi\right)^{1/2}}{\|\hat{g}\|_2},$$

where the square of the denominator is the energy of $g(t)$. Two sinusoids will be discriminated in frequency only if they are more than $\Delta\xi$ apart.[7] Thus $\Delta\xi$ is referred to as the *frequency resolution* of the short-time Fourier transform analysis with window g. Similarly, define the "time spread" Δt of $g(t)$ by

$$\Delta t = \frac{\left(\int_{-\infty}^{\infty} t^2 |g(t)|^2\,dt\right)^{1/2}}{\|g\|_2},$$

where once again the square of the denominator is the energy of $g(t)$. Two pulses will be discriminated in time only if they are more than Δt apart.[7] So, Δt is referred to as the *time resolution* of this same analysis with window g. Now, by the Heisenberg uncertainty principle, the product $\Delta t\Delta\xi$ of time resolution and frequency resolution is bounded below, so one may attain either arbitrarily good time or frequency resolution, but not both. Once the selection of g has been made, the resolution over all time and frequency is *fixed.* Therefore, an alternative to the Gabor transform is necessary if we desire varying levels of resolution in the analysis of a given signal. Recently, wavelet

analysis has been presented as one such alternative. So, let us introduce at this point the idea of a "wavelet":

We shall speak of a function ψ in $L^1(\mathbb{R})$ as being *normalized* and *wavelike* if

$$(i) \quad \psi \in L^2(\mathbb{R}), \qquad (ii) \quad \int_0^\infty |\hat{\psi}(t\xi)|^2 \, \frac{dt}{t} = 1 \quad (\xi \in \mathbb{R}). \tag{4.2}$$

By "wavelike," we are actually referring to the characteristic that ψ has average zero; that is

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(x)\, dx = 0.$$

But $\psi \in L^1(\mathbb{R})$ implies that $\hat{\psi}$ is continuous, so the existence of the integral in (4.2)(ii) is enough to ensure that $\hat{\psi}(0) = 0$. This prompts the following definition.

(4.3) DEFINITION. *When ψ is a normalized, wavelike function in $L^1(\mathbb{R})$, the mapping*

$$f \longrightarrow T_\psi f(r, x) = \frac{1}{r^{1/2}} \int_{-\infty}^{\infty} f(v) \overline{\psi\left(\frac{v-x}{r}\right)}\, dv \tag{4.3)(i}$$

is called the ***continuous wavelet transform,*** *while*

$$T_\psi f \longrightarrow f(v) = \int_0^\infty \int_{-\infty}^{\infty} \frac{1}{r^{1/2}} T_\psi f(r, x) \psi\left(\frac{v-x}{r}\right) \frac{dx dr}{r^2} \tag{4.3)(ii}$$

is called the ***inverse continuous wavelet transform.***

The critical idea here is this: We devise a ψ with wavelike properties (fulfilling a role much like the complex exponential in the Gabor transform), but then we integrate translations *and dilations* of this function against the signal to be analyzed. Resultingly, we obtain *varying levels of resolution* as r ranges over $\mathbb{R}$.

Now, the Gabor transform and its properties are very well known. One very important idea associated with it is the **Wigner-Ville distribution**. Define the "voice transform" $V_{f,g}$ to be

$$V_{f,g}(p, q) = \int_{-\infty}^{\infty} f(v + p)\overline{g(v)} e^{2\pi i (vq + pq/2)}\, dv. \tag{4.4}$$

By a simple change of variables, we observe

$$V_{f,g}(p, q) = \int_{-\infty}^{\infty} f(v + p)\overline{g(v)} e^{2\pi i (v + p/2) q}\, dv$$

$$= \int_{-\infty}^{\infty} f(v + \frac{p}{2})\overline{g(v - \frac{p}{2})} e^{2\pi i v q} \, dv, \tag{4.5}$$

which is also referred to as the **cross ambiguity function of** f **and** g. It is worth noting here the relationship between $V_{f,g}$ and $S_g f$:

$$V_{f,g}(p,q) = \int_{-\infty}^{\infty} f(v+p)\overline{g(v)} e^{2\pi i (v+p/2)q} \, dv$$

$$= \int_{-\infty}^{\infty} f(v)\overline{g(v-p)} e^{2\pi i (v-p/2)q} \, dv = e^{-\pi i p q} \int_{-\infty}^{\infty} f(v)\overline{g(v-p)} e^{2\pi i v q} \, dv$$

$$= e^{-\pi i p q} S_g f(p, -q).$$

We obtain the Wigner-Ville distribution, $W_{f,g}$, by taking the two-dimensional Fourier transform of $V_{f,g}$ [8]:

$$W_{f,g}(x,\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_{f,g}(p,q) e^{-2\pi i (qx + p\xi)} \, dq dp$$

$$= \int_{-\infty}^{\infty} f(x + \frac{p}{2})\overline{g(x - \frac{p}{2})} e^{-2\pi i p \xi} \, dp. \tag{4.6}$$

This distribution has been studied extensively and has served as the prototype of all time-frequency distributions.[9] Our goal is to carry over this idea to the wavelet transform. We take the following approach:

Represent the **Heisenberg group,** $\mathcal{H}$**,** associated with $\mathbb{R}$ as the group of matrices of the following form

$$\mathcal{H} = \left\{ \begin{bmatrix} 1 & x & \xi & 2s \\ 0 & 1 & 0 & \xi \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} : x, \xi, s \in \mathbb{R} \right\} . \tag{4.7}$$

For ease of notation, we express the group elements by

$$(x,\xi,s) \sim \begin{bmatrix} 1 & x & \xi & 2s \\ 0 & 1 & 0 & \xi \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{4.8}$$

so matrix multiplication corresponds to the group operation

$$(x,\xi,s).(y,\eta,t) = (x+y, \xi+\eta, s+t+\tfrac{1}{2}(x\eta - y\xi)) . \tag{4.9}$$

It is quickly verified that $\mathcal{H}$ is in fact a group, with element inverses given by

$$(x, \xi, s)^{-1} = (-x, -\xi, -s). \tag{4.10}$$

Now we introduce the Schrödinger representation π of $\mathcal{H}$, which is a representation extending to $\mathcal{H}$ the basic operations

$$\pi(x, 0, 0) : f(v) \longrightarrow f(v + x) \qquad (x \in \mathbb{R}), \tag{4.11}$$

and

$$\pi(0, \xi, 0) : f(v) \longrightarrow e^{2\pi i v\xi} f(v) \qquad (\xi \in \mathbb{R}). \tag{4.12}$$

These are the operations of translation and modulation which underlie the Euclidean Fourier transform. Now, since

$$(x, 0, 0).(0, \xi, 0).(x, 0, 0)^{-1}.(0, \xi, 0)^{-1} = (0, 0, x\xi)\ ,$$

conditions (4.11) and (4.12) completely determine π on the subgroup $Z = \{(0, 0, s) : s \in \mathbb{R}\}$ of $\mathcal{H}$; indeed, a simple calculation shows that

$$\pi(0, 0, x\xi) : f(v) \longrightarrow e^{2\pi i x\xi} f(v)\ ,$$

and so, in general,

$$\begin{aligned}\pi(x, \xi, s) f(v) &= \pi\Big(0, \xi, s + \tfrac{1}{2}x\xi\Big)\pi(x, 0, 0) f(v) \\ &= \pi\Big(0, 0, s + \tfrac{1}{2}x\xi\Big)\pi(0, \xi, 0) f(v + x) = e^{2\pi i s} e^{2\pi i (v\xi + \frac{1}{2}x\xi)} f(v + x).\end{aligned} \tag{4.13}$$

Notice that this representation is well-suited to express the Voice transform, since

$$\begin{aligned}V_{f,g}(p, q) &= \int_{-\infty}^{\infty} f(v + p)\overline{g(v)} e^{2\pi i (vq + pq/2)}\, dv \\ &= \int_{-\infty}^{\infty} \pi(p, q, 0) f(v)\overline{g(v)}\, dv.\end{aligned} \tag{4.14}$$

Likewise, the Gabor transform may be represented by

$$S_g f(p, -q) = e^{\pi i pq} V_{f,g}(p, q) = \int_{-\infty}^{\infty} \pi(p, q, \frac{pq}{2}) f(v)\overline{g(v)}\, dv. \tag{4.15}$$

In the very same fashion, we represent the **affine group,** $\mathcal{A}$**,** associated with $\mathbb{R}$ as a group of matrices of the following form

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & x & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 1/r & -x/r \\ 0 & 0 & 0 & 1 \end{bmatrix} : x, r \in \mathbb{R}, r > 0 \right\}\ . \tag{4.16}$$

Again, we simplify notation by expressing the group elements by

$$(x, r) \sim \begin{bmatrix} 1 & x & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 1/r & -x/r \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{4.17}$$

The more common (and indeed simpler) way to represent this group is with 2×2 matrices, so that

$$(x, r) \sim \begin{bmatrix} 1 & x \\ 0 & r \end{bmatrix} ,$$

but it is a simple task to show that these are equivalent, and the form in Eq. (4.17) is more appropriate for our purposes here. Matrix multiplication corresponds to the group operation

$$(x, r).(y, \rho) = (\rho x + y, r\rho). \tag{4.18}$$

The element inverses in $\mathcal{A}$ are given by

$$(x, r)^{-1} = (-x/r, 1/r). \tag{4.19}$$

Let us now incorporate the Heisenberg and affine groups into a single representation, and expand the π representation so as to include the wavelet transform. Introduce the **affine-Heisenberg group, $\mathcal{G}$,** of which both the affine and Heisenberg groups are subgroups by letting

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & x & \xi & 2s \\ 0 & r & 0 & r\xi \\ 0 & 0 & 1/r & -x/r \\ 0 & 0 & 0 & 1 \end{bmatrix} : x, \xi, s, r \in \mathbb{R} \, , \; r > 0 \right\} . \tag{4.20}$$

Once more, it will be notationally convenient to write (x, ξ, s, r) for an element of $\mathcal{G}$ instead of its matrix realization. With this notation, the group operation on $\mathcal{G}$ becomes

$$(x, \xi, s, r).(y, \eta, t, \rho) = \Big(\rho x + y, \;\; \xi/\rho + \eta, \;\; s + t + \tfrac{1}{2}(\rho x\eta - y\xi/\rho), \;\; r\rho\Big). \tag{4.21}$$

We extend the π representation from the Heisenberg group to $\mathcal{G}$ by the operation

$$\pi(0, 0, 0, r) : f(v) \longrightarrow r^{1/2} f(rv) \qquad (r > 0). \tag{4.22}$$

Consequently, when we combine Eq. (4.22) with Eqs. (4.11) and (4.12), we determine π on all of $\mathcal{G}$ to be

$$\pi(x, \xi, s, r) : f(v) \longrightarrow r^{1/2} e^{2\pi i s} e^{2\pi i (rv\xi + \frac{1}{2}x\xi)} f(rv + x) \, . \tag{4.23}$$

To verify that π is in fact a well-defined representation on $\mathcal{G}$, observe that

$$\pi(x,\xi,s,r)\Big(\pi(y,\eta,t,\rho)f(v)\Big) = \pi(x,\xi,s,r)\rho^{1/2}e^{2\pi it}e^{2\pi i(\rho v\eta+\frac{1}{2}y\eta)}f(\rho v+y)$$

$$= r^{1/2}\rho^{1/2}e^{2\pi i(s+t)}e^{2\pi i(rv\xi+\frac{1}{2}x\xi)}e^{2\pi i(\rho(rv+x)\eta+\frac{1}{2}y\eta)}f(\rho(rv+x)+y).$$

On the other hand,

$$\pi\Big((x,\xi,s,r).(y,\eta,t,\rho)\Big)f(v)$$

$$= \pi\Big(\rho x+y,\ \xi/\rho+\eta,\ s+t+\tfrac{1}{2}(\rho x\eta - y\xi/\rho),\ r\rho\Big)f(v)$$

$$= (r\rho)^{1/2}e^{2\pi i(s+t+\frac{1}{2}(\rho x\eta-y\xi/\rho))}e^{2\pi i(r\rho v(\xi/\rho+\eta)+\frac{1}{2}(\rho x+y)(\xi/\rho+\eta))}f(r\rho v+\rho x+y)$$

$$= r^{1/2}\rho^{1/2}e^{2\pi i(s+t)}e^{2\pi i(rv\xi+\frac{1}{2}x\xi)}e^{2\pi i(\rho(rv+x)\eta+\frac{1}{2}y\eta)}f(\rho(rv+x)+y).$$

Hence, $\pi(x,\xi,s,r)\Big(\pi(y,\eta,t,\rho)f(v)\Big) = \pi\Big((x,\xi,s,r).(y,\eta,t,\rho)\Big)f(v)$.

Now we finally have a representation which puts the Gabor and wavelet transforms "under one roof," so to speak. Observe that in the case $s = \frac{pq}{2}$ and $r = 1$, $\mathcal{G}$ reduces to the Heisenberg group and by Eq. (4.15) the Gabor transform is represented as

$$S_g f(p,-q) = \int_{-\infty}^{\infty} \pi(p,q,\frac{pq}{2},0)f(v)\overline{g(v)}\,dv.$$

On the other hand, letting $\xi = 0$ and $s = 0$, we obtain

$$\int_{-\infty}^{\infty} \pi(x,0,0,r)f(v)\overline{\psi(v)}\,dv = r^{1/2}\int_{-\infty}^{\infty} f(rv+x)\overline{\psi(v)}\,dv$$

$$= \frac{1}{r^{1/2}}\int_{-\infty}^{\infty} f(v)\overline{\psi\Big(\frac{v-x}{r}\Big)}\,dv = T_\psi f(r,x),$$

which gives us a representation of the wavelet transform.

So, let us refer back to the Wigner-Ville distribution and try to find some analogous setting in the wavelet domain. Recall that

$$W_{f,g}(x,\xi) = \mathcal{F}\Big(V_{f,g}(p,q)\Big) = \mathcal{F}\Big(\int_{-\infty}^{\infty} \pi(p,q,0,0)f(v)\overline{g(v)}\,dv\Big),$$

where $\mathcal{F}$ here denotes the two-dimensional Fourier transform on $\mathbb{R}^2$. In this case, $V_{f,g}$ takes signal f and window g to a function on the plane $\{(p,q) : p,q \in \mathbb{R}\}$. The group structure with regard to both variables p and q is *additive*, hence $\mathcal{F}$ is the "natural" transform to apply to $V_{f,g}$.

Now consider

$$T_\psi f(r,x) = \int_{-\infty}^{\infty} \pi(x,0,0,r)f(v)\overline{\psi(v)}\,dv.$$

Here $T_\psi f$ takes signal f and wavelet ψ to a function on the upper half-plane $\{(r,x): r,x \in \mathbb{R}, r>0\}$. But we observe that r plays a *scaling* role here, and hence the group structure with regard to this variable is *multiplicative.* As we have seen, the action of the Mellin transform on the multiplicative group is analogous to that of the Fourier transform on the additive group. This leads us to consider the effects of the Mellin transform on the scale variable in $T_\psi f$ (we will utilize both $\mathfrak{M}$ and $\mathcal{M}_+$ here):

Let f and ψ be in $H^2_+(\mathbb{R})$. Then

$$\begin{aligned}\mathfrak{M}: T_\psi f \longrightarrow &\int_0^\infty \Big(\int_{-\infty}^\infty \pi(x,0,0,r) f(v)\overline{\psi(v)}\,dv\Big) r^{-2\pi i t-1}\,dr \\ &= \int_0^\infty \Big(\int_{-\infty}^\infty r^{1/2} f(rv+x)\overline{\psi(v)}\,dv\Big) r^{-2\pi i t-1}\,dr.\end{aligned}$$

By Parseval's theorem, and since $f, \psi \in H^2_+(\mathbb{R})$,

$$\begin{aligned}&\int_0^\infty \Big(\int_{-\infty}^\infty r^{1/2} f(rv+x)\overline{\psi(v)}\,dv\Big) r^{-2\pi i t-1}\,dr \\ &= \int_0^\infty \Big(\int_0^\infty r^{-1/2} e^{2\pi i x(\xi/r)} \hat{f}\Big(\frac{\xi}{r}\Big)\overline{\hat{\psi}(\xi)}\,d\xi\Big) r^{-2\pi i t}\,\frac{dr}{r}.\end{aligned}$$

Now we change the order of integration and make the variable change $u=\xi/r$ to obtain

$$\begin{aligned}&\int_0^\infty \Big(\int_0^\infty r^{-1/2} e^{2\pi i x(\xi/r)} \hat{f}\Big(\frac{\xi}{r}\Big)\overline{\hat{\psi}(\xi)}\,d\xi\Big) r^{-2\pi i t}\,\frac{dr}{r} \\ &= \int_0^\infty \Big(\int_0^\infty e^{2\pi i x u}\Big(\frac{u}{\xi}\Big)^{1/2} \hat{f}(u)\Big(\frac{u}{\xi}\Big)^{2\pi i t}\frac{du}{u}\Big)\overline{\hat{\psi}(\xi)}\,d\xi \\ &= \Big(\int_0^\infty \overline{\hat{\psi}(\xi)}\xi^{-2\pi i t-1/2}\,d\xi\Big)\Big(\int_0^\infty e^{2\pi i x u}\hat{f}(u) u^{2\pi i t-1/2}\,du\Big) \\ &= \overline{\mathcal{K}\psi}(t)\Big(\mathcal{T}_x * \mathcal{K}f\Big)(t)\end{aligned}$$

(since $\mathcal{K} = \mathcal{M}_+ \circ \mathcal{F}$ for functions in $H^2_+(\mathbb{R})$). Thus,

$$\mathfrak{M}: T_\psi f \longrightarrow (\overline{\mathcal{K}\psi})(\mathcal{T}_x * \mathcal{K}f). \tag{4.24}$$

This result is of particular interest for at least two reasons. First, notice that taking the Mellin transform with respect to the scale variable in $T_\psi f$ allows us to "factor out" the wavelet transform, i.e., $\mathfrak{M}(T_\psi f)$ is a product of a transform of ψ with a transform of f. Second, Eq. (4.24) allows a representation of $T_\psi f$ which is entirely analogous to the standard representation of the Wigner-Ville distribution. Indeed, taking the Fourier transform of the voice transform $V_{f,g}$ with respect to both variables

we obtained the Wigner-Ville distribution, most commonly written, as in Eq. (4.6), as

$$W_{f,g}(x,\xi) = \int_{-\infty}^{\infty} f(x+\frac{p}{2})\overline{g(x-\frac{p}{2})}e^{-2\pi i p\xi}\,dp. \tag{4.25}$$

Analogously, taking the Mellin transform with respect to the scale variable and the Fourier transform with respect to the shift variable of the wavelet transform $T_{\psi}f$ we obtain

$$\begin{aligned}\mathcal{F}\times\mathfrak{M} : T_{\psi}f \longrightarrow & \int_{-\infty}^{\infty} \mathfrak{M}(T_{\psi}f)(t,x)e^{-2\pi i x\xi}\,dx \\ &= \int_{-\infty}^{\infty} (\mathcal{T}_x * \mathcal{K}f)(t)\overline{\mathcal{K}\psi}(t)e^{-2\pi i x\xi}\,dx \\ &= \Big(\int_{-\infty}^{\infty} (\mathcal{T}_x * \mathcal{K}f)(t)e^{-2\pi i x\xi}\,dx\Big)\overline{\mathcal{K}\psi}(t).\end{aligned}$$

Because this mapping is exactly the affine analogue of the Wigner-Ville distribution, we shall call the function

$$\mathcal{W}_{f,\psi}(t,\xi) = \Big(\int_{-\infty}^{\infty} (\mathcal{T}_x * \mathcal{K}f)(t)e^{-2\pi i x\xi}\,dx\Big)\overline{\mathcal{K}\psi}(t) \tag{4.26}$$

the **affine Wigner-Ville distribution of f and ψ.**

Previous attempts at affine versions of the Wigner-Ville distribution have been made by Altes[10], Flandrin [11], and Parks and Shenoy [8], but none utilize the $\mathcal{K}$-transform. What we have done is begin an analysis of the $\mathcal{K}$-transform in a setting which unifies both the Gabor transform and the wavelet transform. Its application to this unified setting, and its suitability to the work of Altes mentioned in the previous section, indicate that the $\mathcal{K}$-transform, as the composition of Fourier and Mellin transforms, is perhaps a worthwhile topic for further investigation. In fact, it appears this transformation may be an underlying element in the realm of wavelets and other such methods of time-scale analysis; as these areas are further pursued so might also the $\mathcal{K}$-transform be more thoroughly considered.

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